Waves in more than one dimension

26TH OCTOBER 2020

Plane wave representation in two and three dimensions

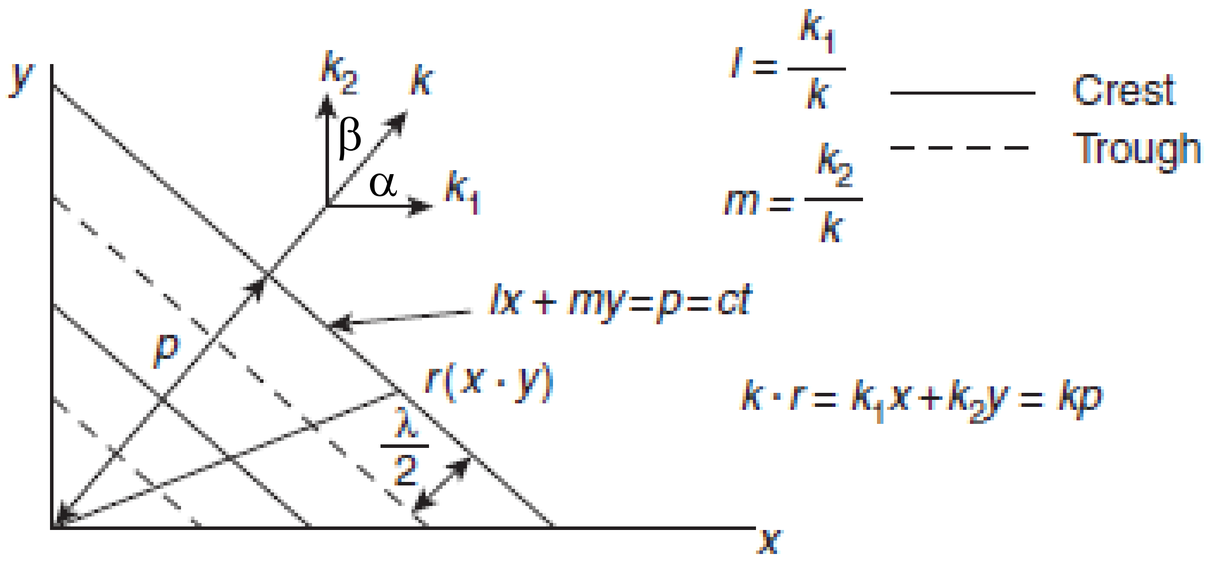
- Basically, the plane wave representation moving in one dimension may be written as

$$E = E_0 e^{i(\omega t - kx)}$$

- Consider the argument representing the phase ;i.e., $\omega t - kx$, the term kx illustrates the **phase shift** due to the wave displacement along the propagation direction where the wave number $k = 2\pi/\lambda$.
- What should this term be modified to if the plane wave propagates in two or three dimensions?
- To modify this term, the **direction cosine** has to be discussed.!

Direction cosine

- By definition, the **direction cosine** or **directional cosine** of a vector is a cosine of the angle between the vector and the three coordinate axes.



- According to the left picture, the direction cosine of the wave vector k are composed of

  cos α = k_1/k and cos β = k_2/k

- Cos α and cos β are represented by *l* and *m*, respectively.

- "*l*x + *m*y = p" represents the line equation in two dimensions where p is the perpendicular distance from the line to the origin.

Modified phase representation

- The phase difference ϕ between the origin and a given line as seen from the figure in previous slide can be written as

$$\phi = \frac{2\pi}{\lambda}\left(\text{path difference}\right) = \frac{2\pi}{\lambda}\, p = \frac{2\pi}{\lambda}\left(lx + my\right)$$

$$= \frac{2\pi}{\lambda}\left(\left(\cos\alpha\right)x + \left(\cos\beta\right)y\right)$$

$$= k_1 x + k_2 y$$

$$= \vec{k}\cdot\vec{r};\ \text{where}\ \ \vec{k} = k_1\hat{i} + k_2\hat{j}\ ,\ \ \vec{r} = x\hat{i} + y\hat{j}$$

$$\text{and}\ \ \ k^2 = k_1^2 + k_2^2$$

$$\therefore \phi = \frac{2\pi}{\lambda}\left(k_1 x + k_2 y\right)$$

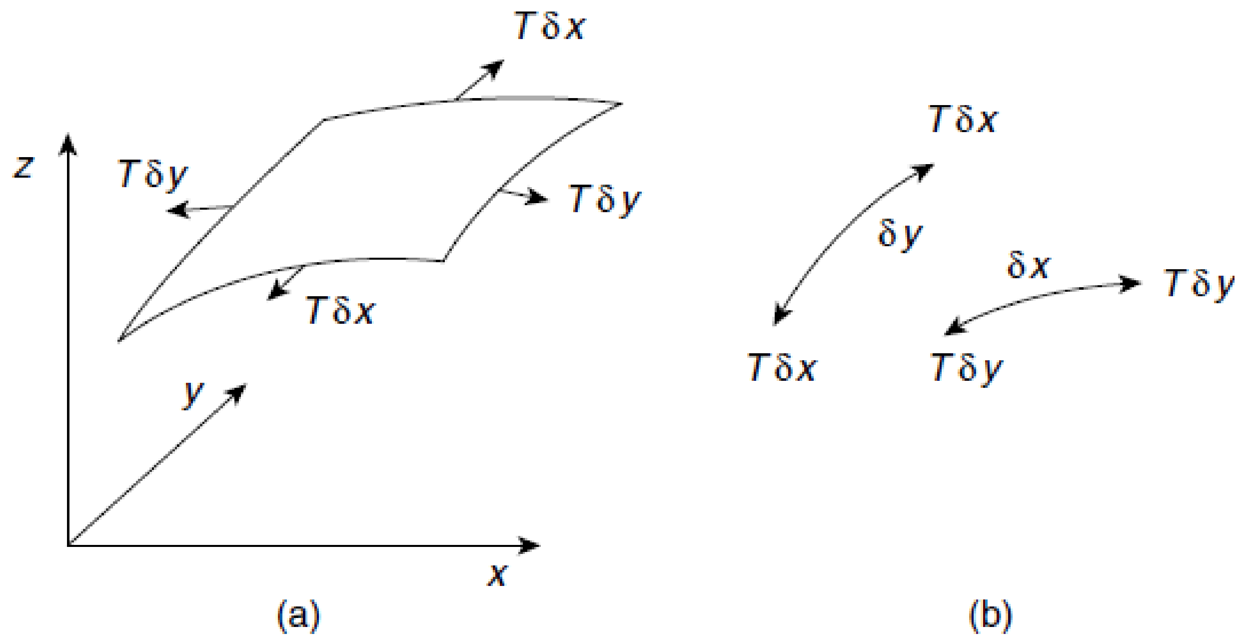
Plane wave representation in 2 and 3 D

- Therefore, the plane wave representation moving in 2 and 3 D can be written as

$$E = E_0 e^{i\left(\omega t - \left(k_1 x + k_2 y\right)\right)}; \quad 2\text{D}$$

$$E = E_0 e^{i\left(\omega t - \left(k_1 x + k_2 y + k_3 z\right)\right)}; \quad 3\text{D}$$

Equation of motion in two dimensions : Rectangular membrane



- Consider the rectangular membrane of a uniform membrane vibrating in the z-direction.
- Area of the membrane = δxδy. The mass of the membrane is given as ρδxδy.
- **Restoring forces** acting on length δx and δy are given as T δy and T δx
- The equation of motion for the membrane is found to be

$$T\delta y\frac{\partial^2 z}{\partial x^2}\delta x+T\delta x\frac{\partial^2 z}{\partial y^2}\delta y=\rho\delta x\delta y\frac{\partial^2 z}{\partial t^2}$$

Wave equation and corresponding wave function in two dimension : Rectangular membrane

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\rho}{T}\frac{\partial^2 z}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}$$

- Wave function that satisfies the wave equation is given as

$$z = Ae^{i\left[\omega t - \left(\vec{k}\cdot\vec{r}\right)\right]} = Ae^{i\left[\omega t - \left(k_1 x + k_2 y\right)\right]}$$

$$\text{where } k^2 = k_1^2 + k_2^2$$

- The wave function represents the plane wave propagating in the xy plane.

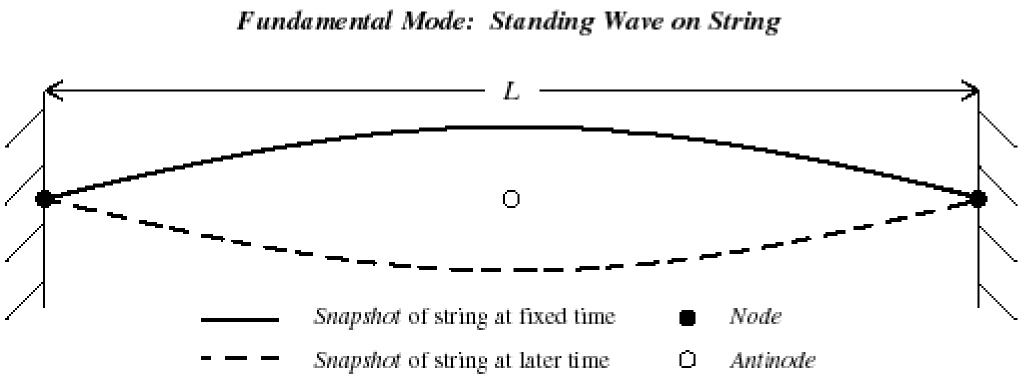
Normal modes and the method of Separation of variables

- What we are interested here is the **wave function of the standing waves on the rectangular membrane**.
- **The standing waves will be called normal modes for the rectangular membrane.**
- This can be found by introducing the boundary conditions at both sides of the rectangular membrane.
- However, to clarify the method, the analytical procedure will begin with one dimensional standing wave happening on the string fixed at both ends.

Wave function for one-dimensional case with boundary conditions

- The boundary conditions for the string fixed at both ends: displacement ϕ = 0 at x = 0 and x = L at all times.

- The wave equation is given as $$\frac{\partial^2\phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$$



- A solution of the equation can be written as the product of two terms
  ϕ = X(x)T(t).
- By using the method of separation of variable,

$$\phi = \begin{cases} \sin kx \cos ckt \\ \sin kx \sin ckt \end{cases}$$

Wave function of standing wave

Problem

Determine **the wave function and the normal modes of vibration** for the standing wave of a string fixed at both ends using the method of separation of variable.

Given that the boundary conditions : $\phi = 0$ at $x = 0$ and $x = L$ at all times.

Wave function for one-dimensional case without boundary conditions

- With the method of separation of variable, the solution of the following wave function

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}$$

are found to be $$\phi = X(x)T(t) = Ae^{\pm ikx}e^{\pm ickt}$$

- A familiar form of the above function may be written as

$$\phi = Ae^{i(ckt-kx)} = Ae^{i(\omega t-kx)}$$

- or $$\phi = A\left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} kx \left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} ckt$$

Wave function for two-dimensional case

- Under this circumstance, the wave equation is given as

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$$

- Now, the function ϕ may be written as the product of X(x)Y(y)T(t).
- With the method of separation of variable, a solution of the differential equation may be written as

$$\phi = Ae^{\pm ik_1x}e^{\pm ik_2y}e^{\pm ickt}\text{; where } k^2 = k_1^2 + k_2^2$$

- Also the solution may be given as $$\phi = A\begin{matrix}\sin\\ \cos\end{matrix}\Big\} k_1x \begin{matrix}\sin\\ \cos\end{matrix}\Big\} k_2y \begin{matrix}\sin\\ \cos\end{matrix}\Big\} ckt$$

Wave function for three-dimensional case

- Under this circumstance, the wave equation is given as

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$$

- Now, the function ϕ may be written as the product of X(x)Y(y)Z(z)T(t).
- With the method of separation of variable, a solution of the differential equation may be written as

$$\phi = Ae^{\pm ik_1x}e^{\pm ik_2y}e^{\pm ik_3z}e^{\pm ickt};\ \text{where } k^2 = k_1^2+k_2^2+k_3^2$$

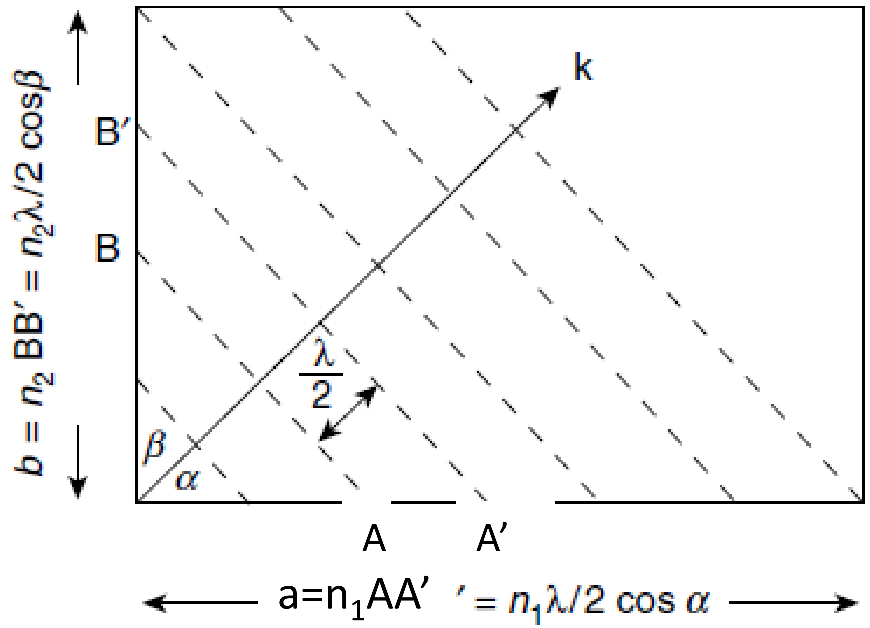
- Also the solution may be given as $$\phi = A\left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} k_1x \left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} k_2y \left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} k_3z \left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} ckt$$

Normal modes in two dimensions on a rectangular membrane

- Consider waves proceed in a direction k on the rectangular membrane of sides *a* and *b*.
- Separation distance between each dotted line is $\lambda/2$.
- Conditions for the existence of standing waves are $a = n_1 AA'$ and $b = n_2 BB'$, where n_1 and n_2 are integers.
- From the figure

$$AA' = \frac{\lambda}{2\cos\alpha} = \frac{\lambda}{2}\frac{k}{k_1} = \frac{\lambda}{2}\frac{2\pi}{\lambda}\frac{1}{k_1} = \frac{\pi}{k_1}$$

From definition of the directional cosine



- From the previous slide, $$k_1 = \frac{n_1\pi}{a}, k_2 = \frac{n_2\pi}{b}$$

- This gives $$k^2 = k_1^2 + k_2^2 = \frac{4\pi^2}{\lambda^2} = \pi^2\left(\frac{n_1^2}{a^2} + \frac{n_2^2}{b^2}\right)$$

- Or $$\frac{2}{\lambda} = \sqrt{\frac{n_1^2}{a^2} + \frac{n_2^2}{b^2}}$$

- The normal mode (vibrating frequency) of the vibrating rectangular membrane can be written as

$$\nu = \frac{c}{2}\sqrt{\frac{n_1^2}{a^2} + \frac{n_2^2}{b^2}};\ \text{where}\ \ c^2 = \frac{T}{\rho}$$

Wave function in two dimensions on a rectangular membrane

- Recall the general wave function in 2 D,

Direction of displacement →

$$z = A \left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} k_1 x \left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} k_2 y \left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} ckt$$

- For a particular wave function that satisfies the rectangular membrane, the boundary conditions have to be considered.
- Boundary conditions : z = 0 at x = 0 and *a*; z =0 at y = 0 and *b*.
- So that

$$z = A\sin\frac{n_1\pi x}{a}\sin\frac{n_2\pi y}{b}\sin ckt$$

Fundamental vibration frequency and conditions for nodal lines

- Recall the general normal mode frequency,

$$\nu = \sqrt{\left(\frac{n_1^2}{a^2} + \frac{n_2^2}{b^2}\right)\frac{T}{4\rho}}$$

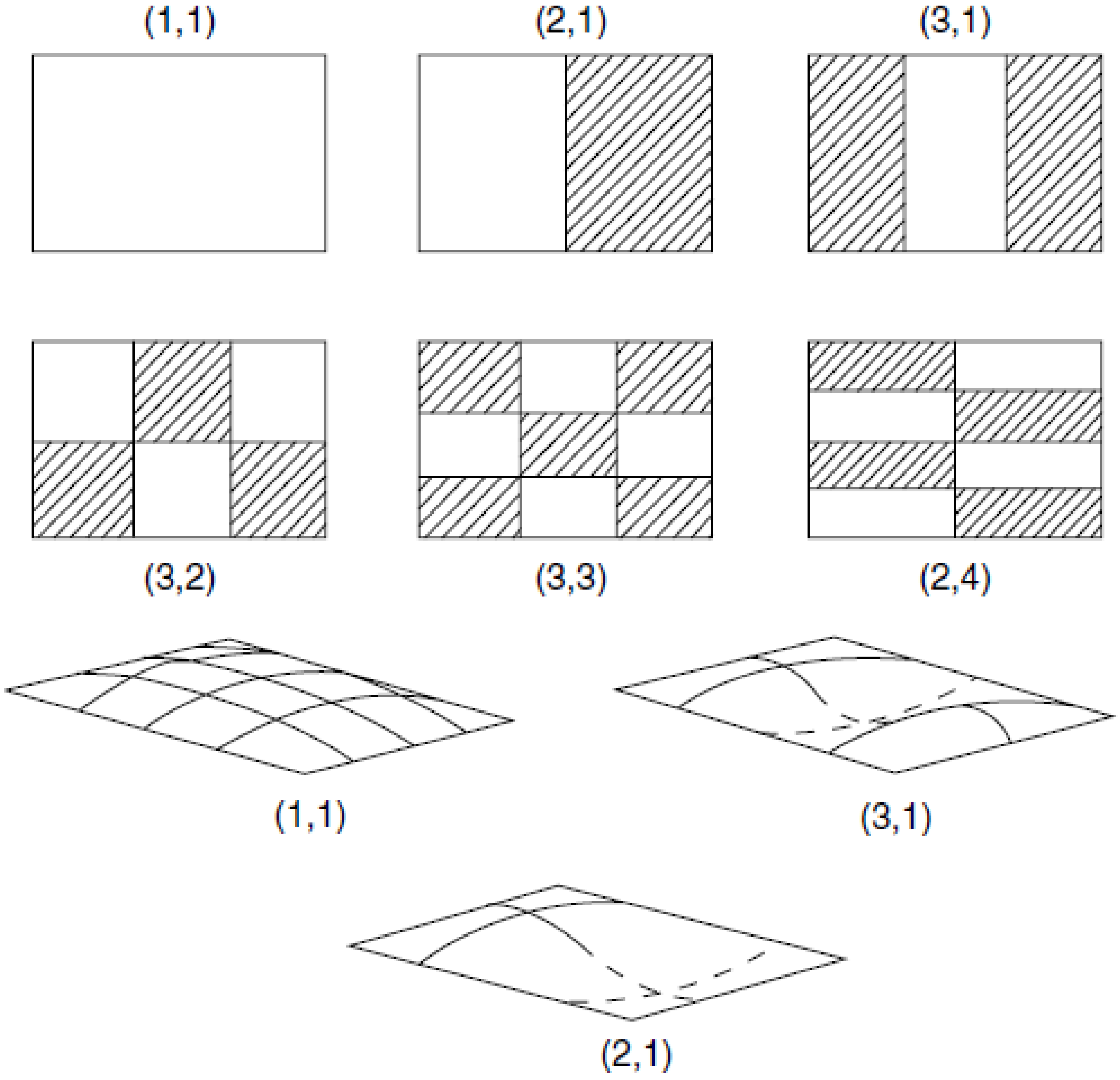
- The fundamental vibration frequency is given by $n_1 = n_2 = 1$,

$$\nu = \sqrt{\left(\frac{1}{a^2} + \frac{1}{b^2}\right)\frac{T}{4\rho}}$$

- Conditions for nodal lines or zero displacement in the general mode $(n_1 n_2)$

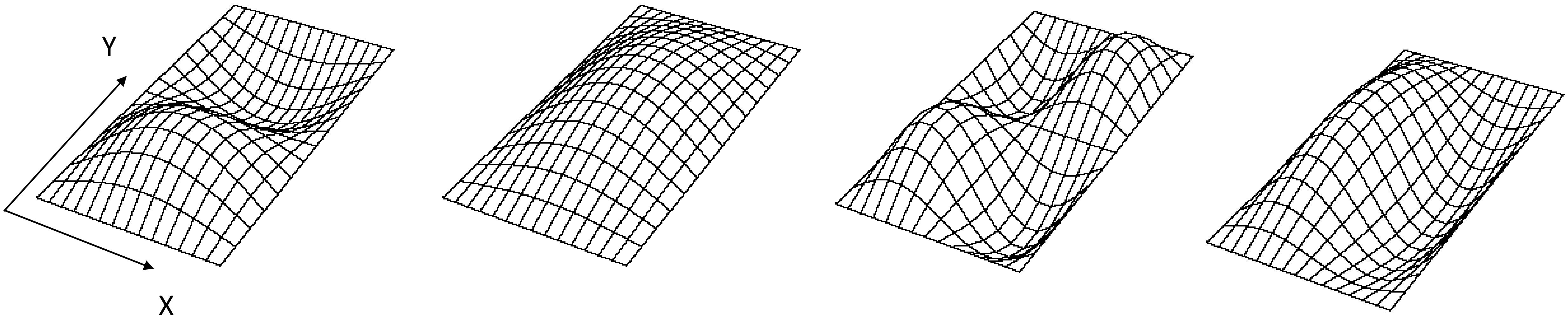
$$x = 0,\ \frac{a}{n_1},\ \frac{2a}{n_1}, \ldots,\ a$$

$$y = 0,\ \frac{b}{n_2},\ \frac{2b}{n_2}, \ldots,\ b$$



Some normal modes on a rectangular membrane

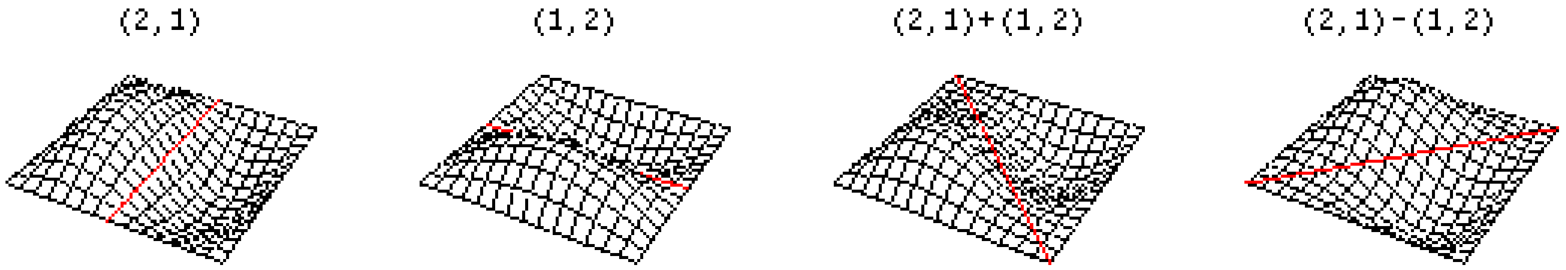
Identify the normal modes on this rectangular membrane



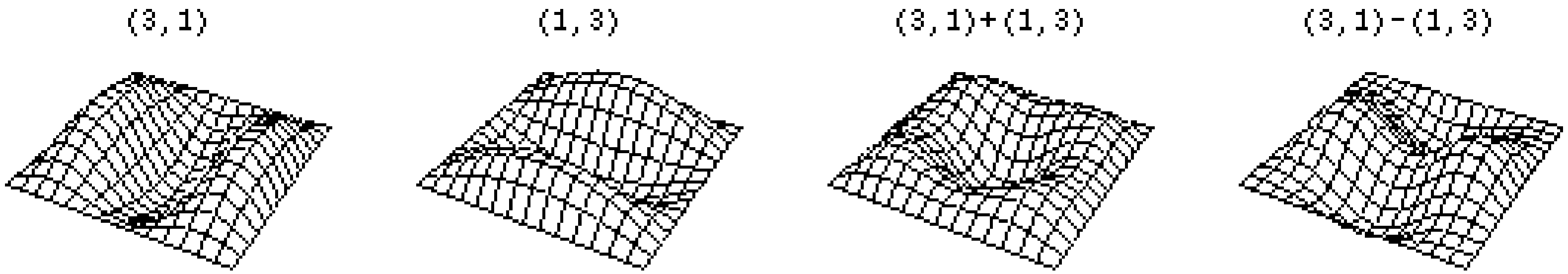
For more animation please visit
https://www.acs.psu.edu/drussell/Demos/rect-membrane/rect-mem.html

Degenerate mode for a square membrane

The (2,1) and (1,2) Modes



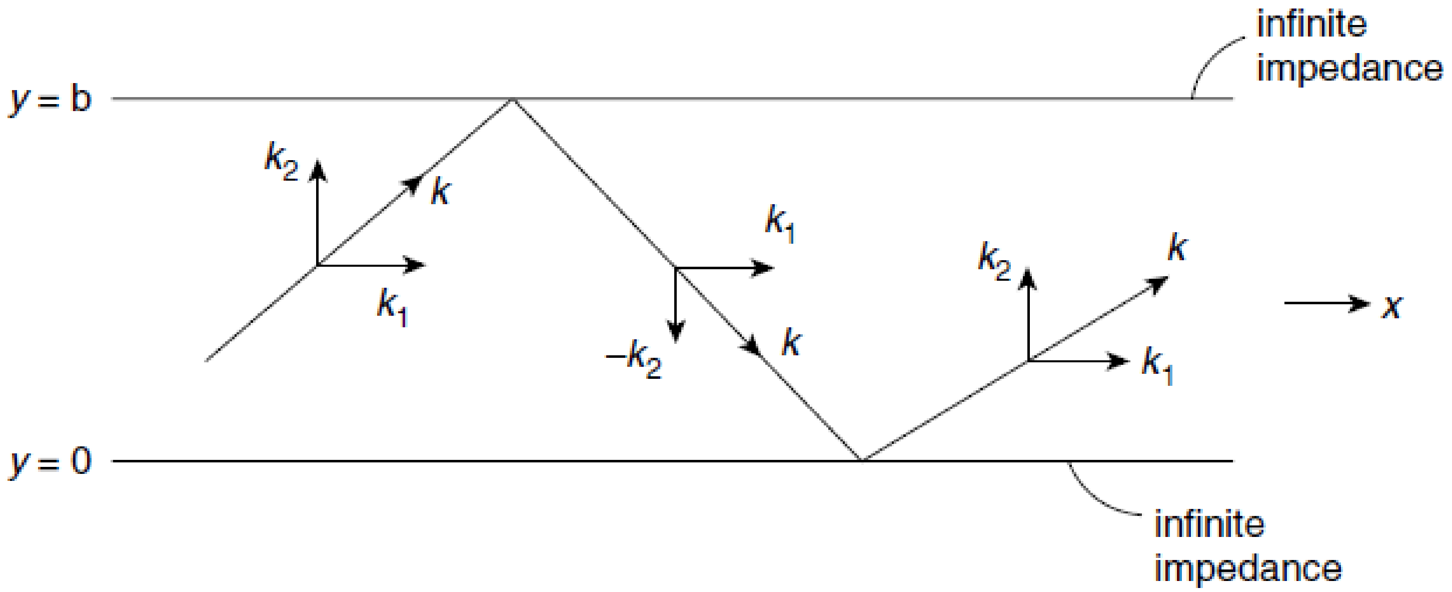
The (3,1) and (1,3) Modes



http://www.acs.psu.edu/drussell/Demos/MembraneSquare/Square.html

Waveguides

- Now, consider a 2D wave propagating in a direction k in the xy plane along a membrane of width *b* stretched under a tension T between two long rigid rods which present and infinite impedance to the wave.



The displacement z is given by the superposition of the incident and reflected waves **(why?)**,

$$z = A_1 e^{i\left[\omega t-\left(k_1 x+k_2 y\right)\right]} + A_2 e^{i\left[\omega t-\left(k_1 x-k_2 y\right)\right]}$$

Displacement of the wave on the membrane

- Recall the superposition from the previous slide, $z = A_1 e^{i[\omega t-(k_1 x+k_2 y)]} + A_2 e^{i[\omega t-(k_1 x-k_2 y)]}$
- Consider the boundary conditions :z = 0 at y= 0 and y =b (the positions of infinite impedance).
- This gives $A_2 = -A_1$ and $\sin k_2 b = 0$ which leads to $k_2 = n\pi/b$.
- Therefore, the **displacement of the wave on the membrane given by the real part becomes**

$z =$

Varying amplitude **Travelling wave**

- **The function represents a wave travelling along the x direction with varying amplitude along y direction.**

What is the phase velocity of the travelling wave?

- Consider the travelling wave on the membrane from previous slide

$$z = +2A_1 \sin k_2 y \sin(\omega t - k_1 x)$$

- The phase velocity of the wave is given as

$$v_p = \frac{\omega}{k_1} = \left(\frac{k}{k_1}\right) v;$$ where v is the velocity of a wave on an infinitely membrane

- Because $k^2 = k_1^2 + k_2^2$, this can be rewritten as $k_1^2 = k^2 - \frac{n^2\pi^2}{b^2}$

- Since k_1 must be real for the wave to propagate, thus $k^2 \geq \frac{n^2\pi^2}{b^2}$

Allowed frequencies of travelling wave

- From the condition $k^2 \geq \dfrac{n^2\pi^2}{b^2}$ and $k = \dfrac{\omega}{\upsilon}$

- The allowed frequencies of travelling wave propagating along the membrane is found to be

$$\nu \geq \frac{n\upsilon}{2b}$$

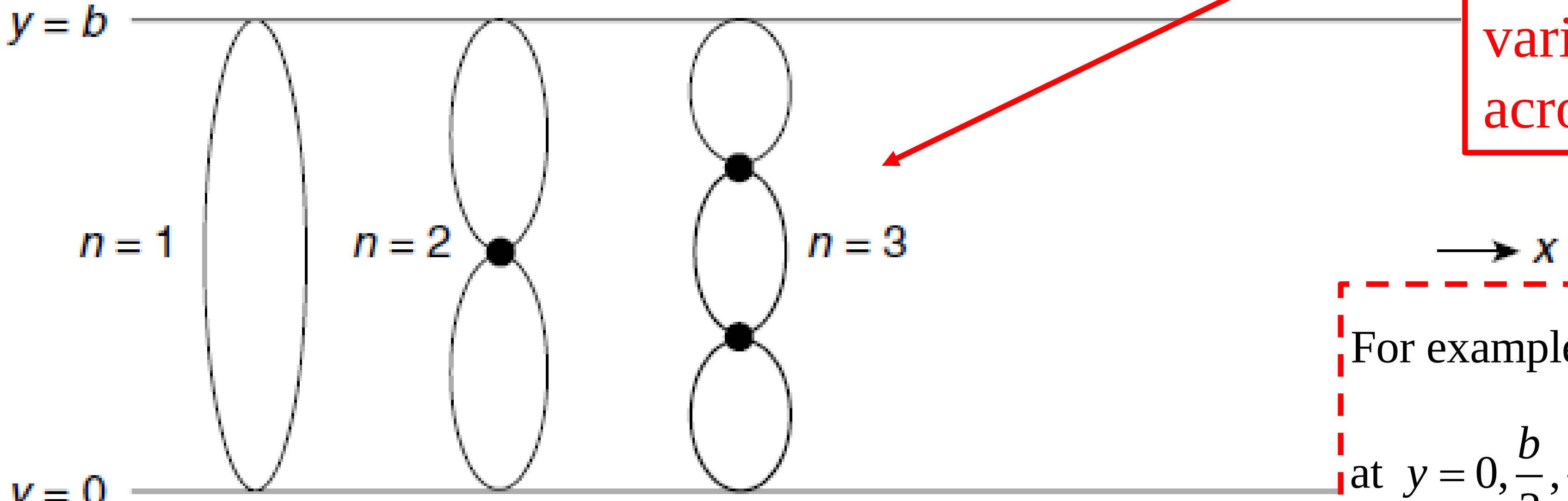
This also represent a cut-off frequency for each mode number *n* indicating that the waveguide acting as a frequency filter.

- Where n defines the mode number in the y direction and the membrane acts as a **waveguide**.

Waveguide membrane

$$z = +2A_1 \sin k_2 y \sin(\omega t - k_1 x)$$

This term expresses the variation of amplitudes across the y direction.

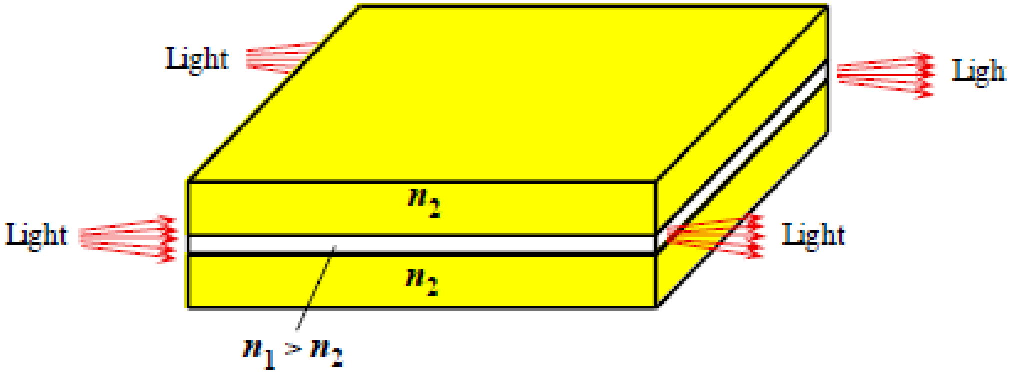


For example; $n = 3, \therefore \sin k_2 y = \sin \frac{3\pi y}{b}$

at $y = 0, \frac{b}{3}, \frac{2b}{3}, b; \sin k_2 y = 0$

Variation of amplitude with y direction for two dimensional wave propagating along the membrane. Normal modes (n = 1, 2 and 3 are shown) are set up along any axis bounded by infinite impedances.

Slab Waveguides



A planar dielectric waveguide has a central rectangular region of higher refractive index n_1 than the surrounding region which has a refractive index n_2. It is assumed that the waveguide is infinitely wide and the central region is of thickness 2 a. It is illuminated at one end by a monochromatic light source.

- Consider a dielectric slab of thickness 2a and refractive index n_1
- Let the dielectric be sandwiched between two semi-infinite regions of index n_2
- Note that $n_2 < n_1$
- The high refractive index is called the core
- The low refractive index is called the cladding
- Only a very thin light beam with a diameter much less than the slab thickness, 2a, will make it into the dielectric slab to reflect off of the cladding.
- The remaining light used to illuminate the structure is "lost"
- Also note that for ease of calculation we will use light that enters the slab waveguide from another medium of index n_1.
- Mode coupling is required to assess the amount of light entering the waveguide from a generic medium of n that will reflect and transmit off the surface of n1 at the front of the slab

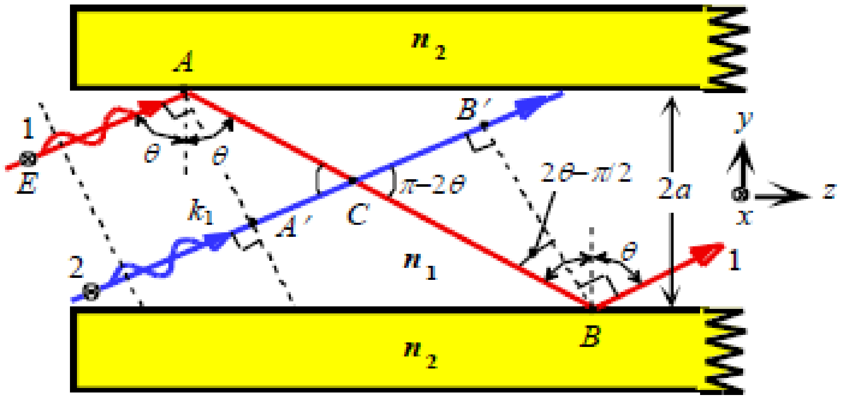
Wave Propagation in Slab Waveguides

- If TIR occurs, then light entering the waveguide easily propagates along in a zigzag fashion
- The zigzag pattern generated by reflection propagates in phase leading to constructive interference within the waveguide
- Light entering the waveguide or reflecting out of phase generates destructive interference and cancels out the propagation amplitude of the EM field.
- Let us suppose that $k1 = kn_1 = 2\pi n_1/\lambda$
- For constructive interference, the phase difference between the two points A and C in the diagram below must be multiples of 2π
- For constructive interference:

$$k_1[2d\cos\theta] - 2\phi = 2\pi m$$

;d = 2a

- Only certain angles of θ and ϕ satisfy this equation for a given integer multiple, m (mode number)
- However, ϕ depends on θ and the polarization state of the incident waves
- Therefore for each m, there will be only 1 allowable θ_m and ϕ_m



Two arbitrary waves 1 and 2 that are initially in phase must remain in after reflections. Otherwise the two will interfere destructively and ca other.

Waveguide Condition

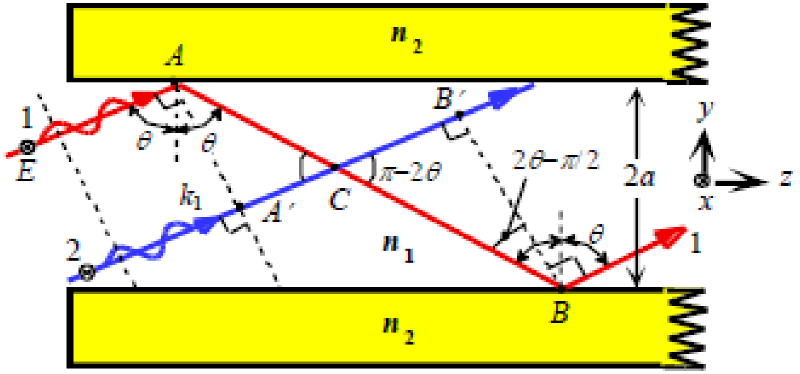
- If we divide the equation for constructive interference by 2 and rewrite, then we have the waveguide condition:

$$\frac{2\pi n_1(2a)}{\lambda}\cos\theta_m - \phi_m = \pi m$$

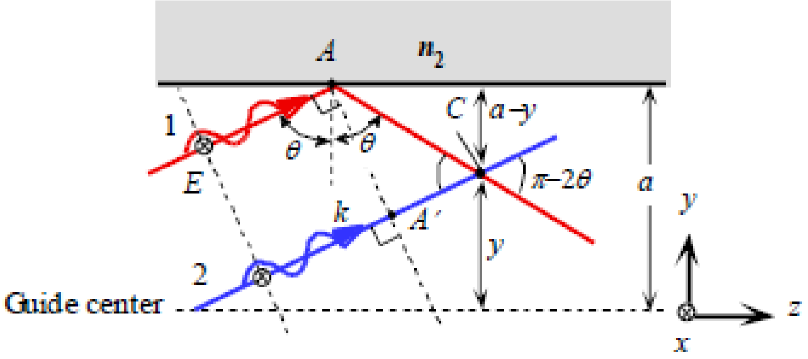
- Where the constructed phase of the wave packet, ϕm is a function of the incidence angle, θm
- This condition is generic for different waveguide shapes, incidence angles, and incident wavelengths.
- Remember: both rays must initially start in phase with one another and remain so after reflection or they will destructively interfere and prevent propagation
- Field in the waveguide

$$E = 2E_m(y)\cos(\omega t - (k_1 \sin\theta_m)z)$$

- $E_m(y)$ is the mode of propagation

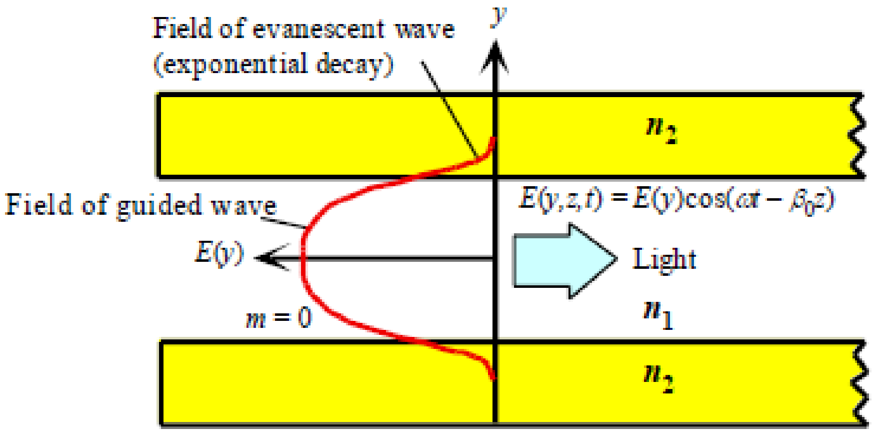


Two arbitrary waves 1 and 2 that are initially in phase must remain in after reflections. Otherwise the two will interfere destructively and ca other.

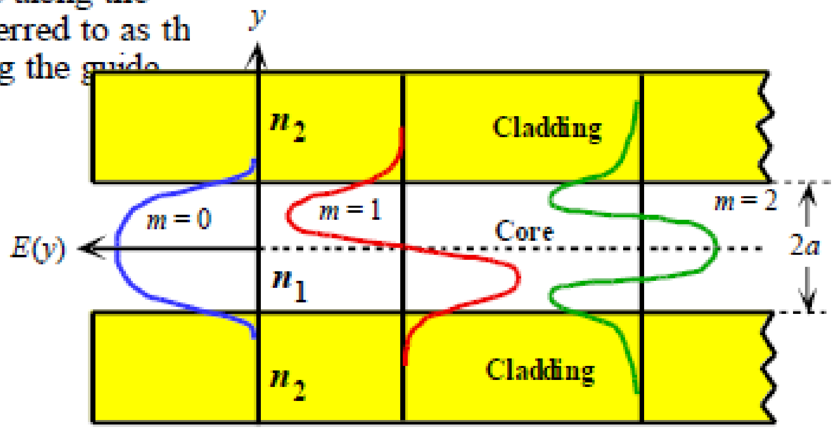


Interference of waves such as 1 and 2 leads to a standing wave pattern along the direction which propagates along z.

Allowed modes in the waveguide



The electric field pattern of the lowest mode traveling wave along the guide. This mode has $m = 0$ and the lowest θ. It is often referred to as th glazing incidence ray. It has the highest phase velocity along the guide



The electric field patterns of the first three modes ($m = 0, 1, 2$) traveling wave along the guide. Notice different extents of field penetration into the cladding.

Single and Multimode Waveguides

- By imposing both TIR and the waveguide condition on the solution for waveguide propagation, we find that only a certain number of modes are allowed in the waveguide
- From

$$\frac{2\pi n_1(2a)}{\lambda}\cos\theta_m - \phi_m = \pi m$$

  we can find an expression for $\sin(\theta_m)$
- Applying the TIR condition,

$$\sin\theta_m > \sin\theta_c$$

- The mode number, m, must satisfy

$$m \le \frac{(2V-\phi)}{\pi}$$

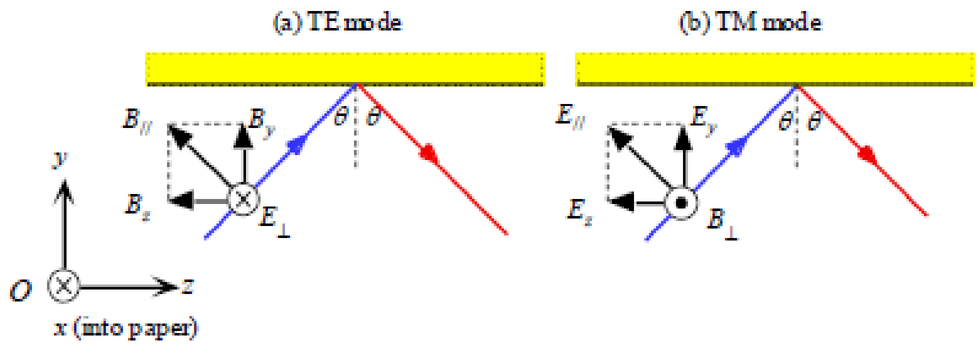
- The V-number, V, also called the normalized thickness or normalized frequency is defined by

$$V = \frac{2\pi a}{\lambda}\sqrt{n_1^2 - n_2^2}$$

- Note: the term thickness is more common for planer waveguides
- The 2a in the term refers to the waveguide geometry, and thus will change with the shape of the waveguide
- Question how does one get V such that only a single mode of propagation exists?
  - At grazing incidence $\theta_m = 90^{\circ}$ and $\phi_m = \pi$
  - Solving for V as a function of m
  - At $V < \pi/2$ only the m=0 mode propagates
  - At $V = \pi/2$ gives the free space cut-off wavelength. Above this wavelength, only the single mode propagation exists

TE and TM Modes

- All discussion up to now have assumed a propagating wave
- However we have two types of propagating waves that generate different phase changes upon reflection and refraction
- So let us now consider TE modes perpendicular to the cross section of the slab: $E_{\perp}=E_x$
- And TM modes parallel to the cross section of the slab: $E_{||} = E_y + E_z$
  - It is interesting that Ez exist along the direction of propagation. It is apparent that Ez is a propagating longitudinal electric field. In free space this is IMPOSSIBLE for such a field to exists, however in a waveguide the interference allows such a phenomenon
  - Note that the same occurs for B in the TE mode
- Because the phase change that accompanies TIR depends on polarization yet is negligible for n_1-n_2<<1, the waveguide condition and the cut-off condition can be taken to be identical for both TE and TM



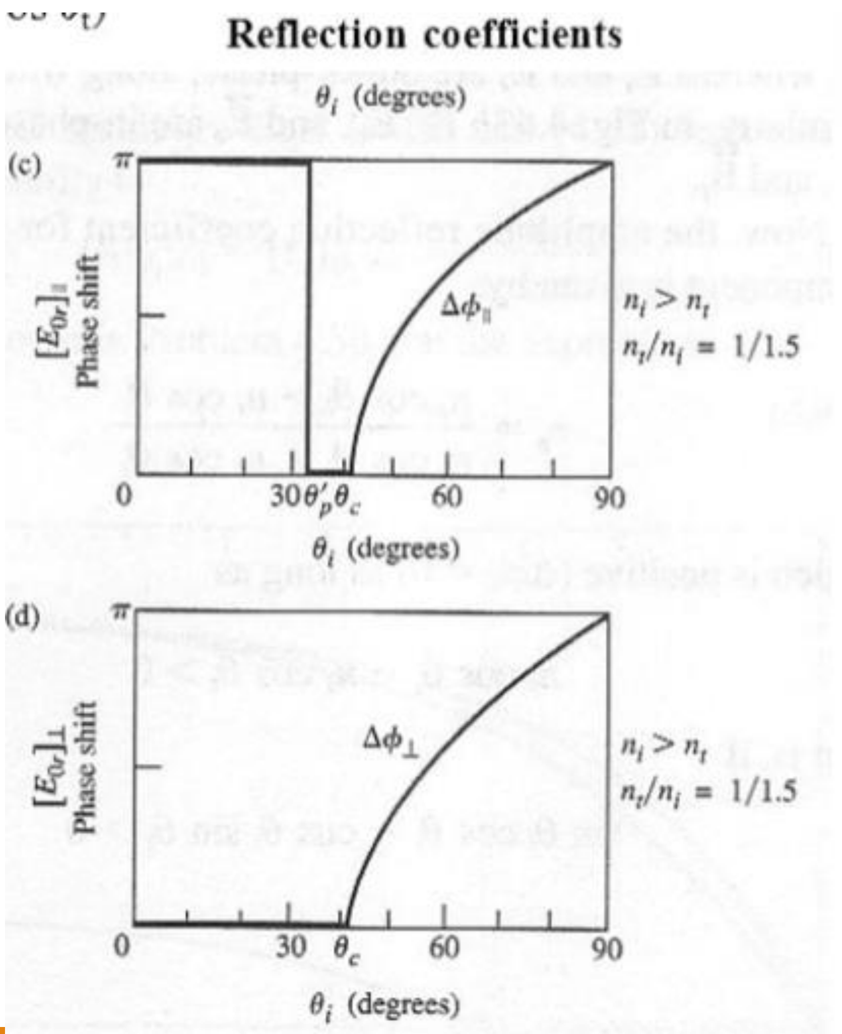
Possible modes can be classified in terms of (a) transelectric field (TE) and (b) transmagnetic field (TM). Plane of incidence is the paper.

Example : a number of modes

Planar dielectric waveguide : a = 50 μm, n_1 = 1.490, n_2 = 1.470 and λ = 1 μm

Determine the **highest mode** m supported by the waveguide and **how many modes** can be supported by the waveguide?

Phase shift due to total internal reflection



$$TM: \tan\left(\frac{\Delta\phi_{\parallel}}{2}\right) = \frac{\sqrt{sin^2\theta_i - n^2}}{n^2 \cos\theta_i}$$

$$TE: \tan\left(\frac{\Delta\phi_{\perp}}{2}\right) = \frac{\sqrt{sin^2\theta_i - n^2}}{\cos\theta_i}$$

$$where \;\; n = \frac{n_2}{n_1}$$

https://slideplayer.com/slide/4645556/

Waveguide Modes

- Planer waveguide: 2a=20um, N1 = 1.455, N2 = 1.440, λ = 900nm (9×10^{-7}m)
- Using waveguide equation and TIR for the TE mode:

$$\tan\left(\frac{1}{2}\phi_m\right) = \frac{\sqrt{\sin^2\theta_m - \left(\frac{n_2}{n_1}\right)^2}}{\cos\theta_m}$$

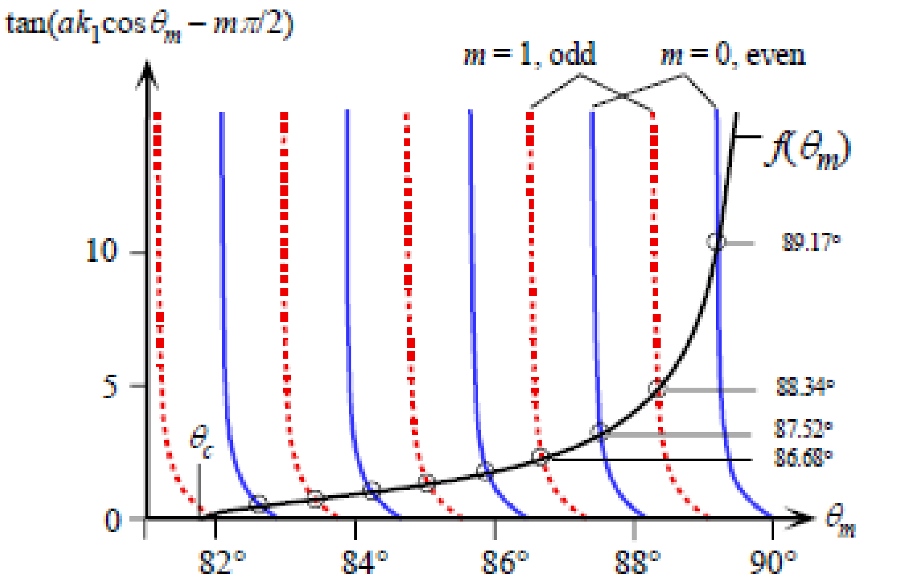
- Using a graphical solution, find the angles for all of the modes.
- Consider:

$$k_1[2a\cos\theta_m] - \phi_m = \pi m$$

$$\tan(ak_1\cos\theta_m - \pi m/2) = \frac{\sqrt{\sin^2\theta_m - \left(\frac{n_2}{n_1}\right)^2}}{\cos\theta_m} = f(\theta)$$

- The left hand side reproduces itself for m = 0,2,4,... and becomes a cot function for odd m

Note: $\theta_c = \arcsin\left(\frac{n_2}{n_1}\right)$



Modes in a planar dielectric waveguide can be determined by plotting the LHS and the RHS of eq. (11).

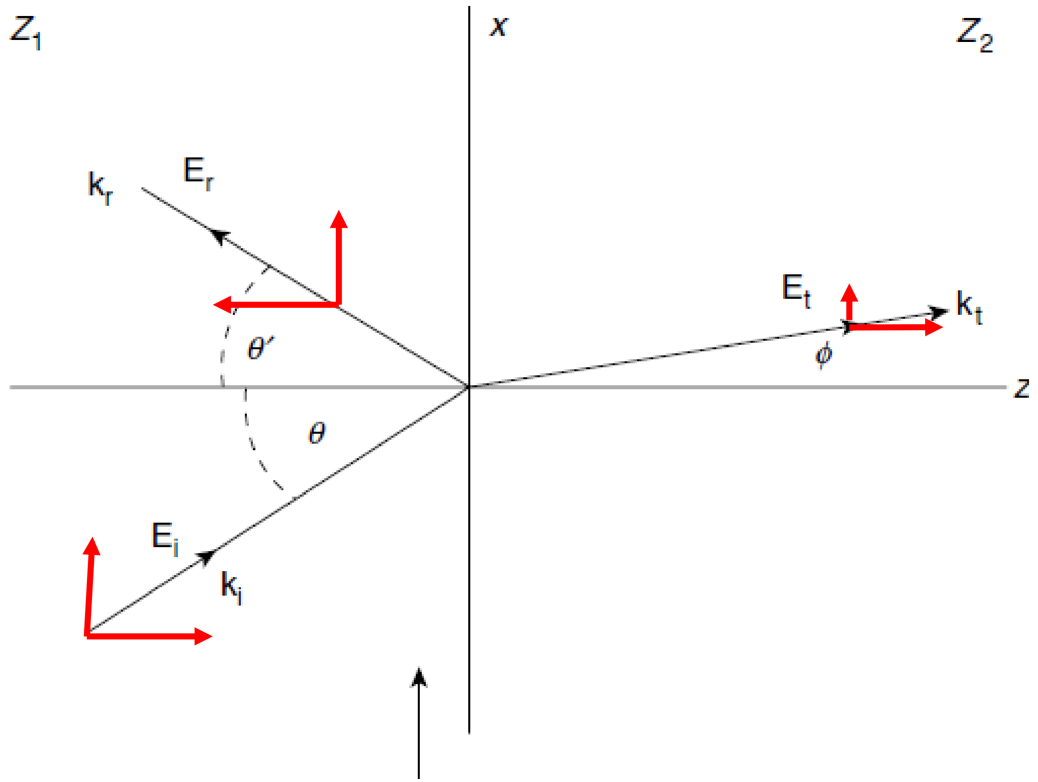
Skin depth of wave into n_2

$$\alpha_m = \frac{2\pi n_2}{\lambda}\sqrt{\left(\frac{n_1}{n_2}\right)^2\sin^2\theta_m - 1}$$

$$\delta_{m=0} = 1/\alpha_{m=0} = 6.91\times10^{-7}\,m$$

$$\delta_{m=9} = 1/\alpha_{m=9} = 38.3\times10^{-7}\,m$$

Reflection and transmission of a three-dimensional wave at a plane boundary



$$E_i = A_i e^{i\left(\omega t - \vec{k}_i \cdot \vec{r}\right)} = A_i e^{i\left[\omega t - k_i\left(x \sin\theta + z\cos\theta\right)\right]}$$

$$E_r = A_r e^{i\left(\omega t - \vec{k}_r \cdot \vec{r}\right)} = A_r e^{i\left[\omega t - k_i\left(x \sin\theta - z\cos\theta\right)\right]}$$

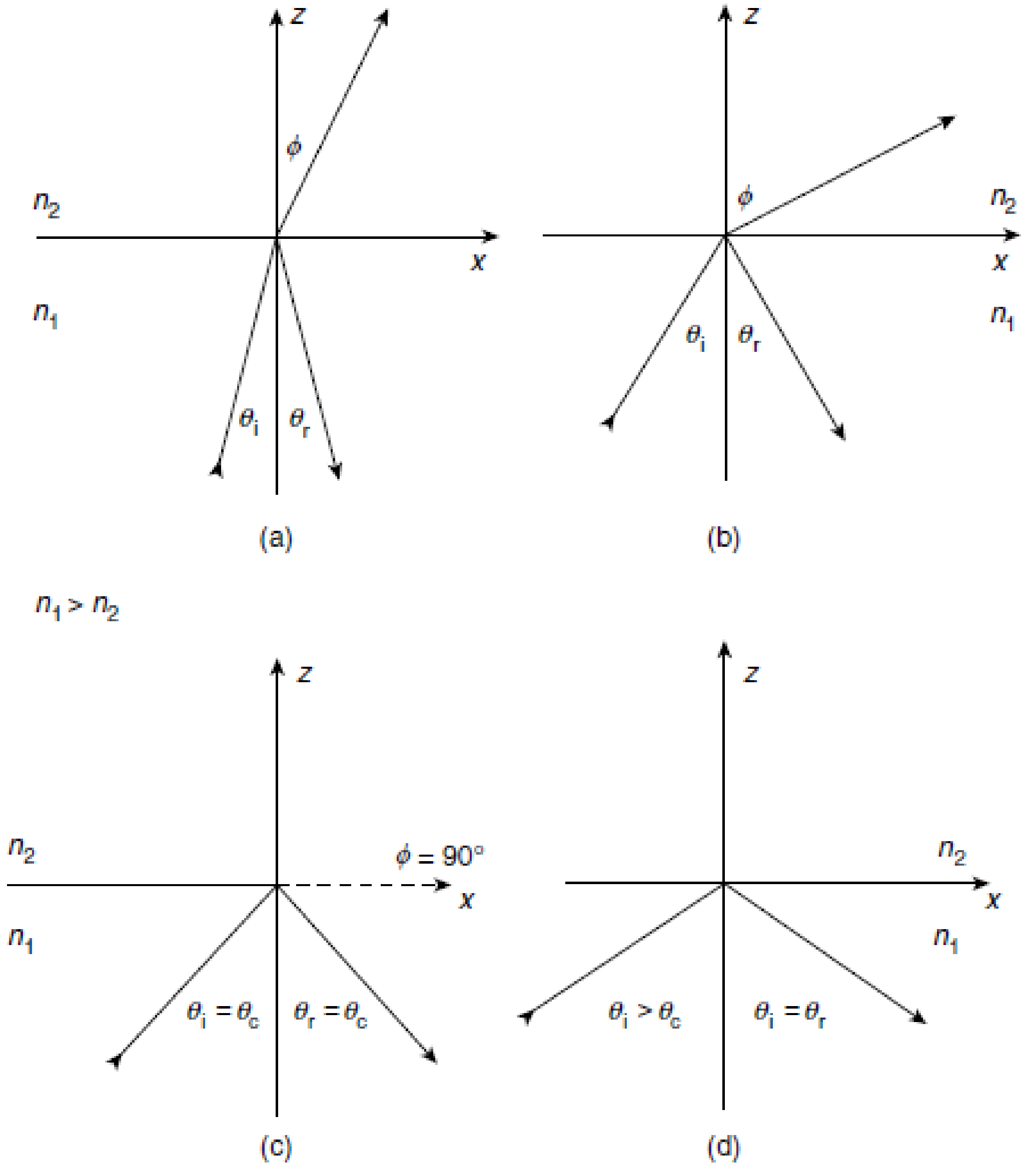
$$E_t = A_t e^{i\left(\omega t - \vec{k}_t \cdot \vec{r}\right)} = A_t e^{i\left[\omega t - k_t\left(x \sin\phi + z\cos\phi\right)\right]}$$

where $\theta = \theta'$, $k_i = k_r$ and $Z_1 > Z_2$

Total internal reflection and evanescent waves

- Consider the propagation of an electromagnetic wave across the boundary between dielectric into air.
- Total internal reflection can take place but boundary conditions still require a transmitted wave known as the **evanescent or surface wave**.
- **The wave propagates in the x direction but its amplitude decays exponentially with z.**

$$\text{crital angle } \theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right) = \sin^{-1} n_r;\ \text{where } \ n_1 > n_2$$



Transmitted wave as the evanescent wave

- Consider in terms of the transmitted electromagnetic wave that satisfies the boundary condition $E_i + E_r = E_t$.

$$E_t = A_t e^{i\left(\omega t - \vec{k}_t \cdot \vec{r}\right)} = A_t e^{i\left[\omega t - k_t\left(x \sin\phi + z\cos\phi\right)\right]}$$

- Because

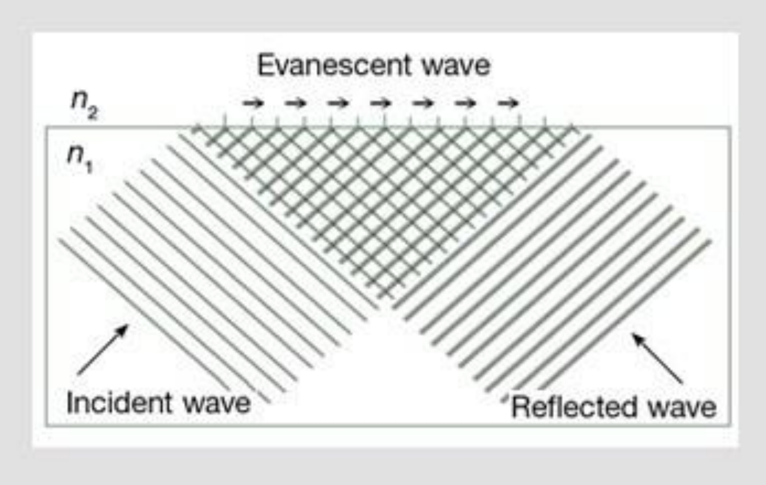
$$\cos^2\phi = 1 - \sin^2\phi = 1 - \sin^2\theta / n_r^2$$

$$\therefore k_t \cos\phi = \pm k_t \left(1 - \sin^2\theta / n_r^2\right)^{\frac{1}{2}}$$

- Which for $\theta > \theta_c$ gives $\sin\theta > n_r$ so that

$$k_t \cos\phi = \pm i k_t \left(\frac{\sin^2\theta}{n_r^2} - 1\right)^{\frac{1}{2}} = \pm i\beta$$

- From the last slide $$k_t \cos\phi = \pm i k_t \left(\frac{\sin^2\theta}{n_r^2} - 1 \right)^{\frac{1}{2}} = \pm i\beta$$



- We also have $$k_t \sin\phi = k_t \sin \theta / n_r$$

- The transmitted wave becomes

$$E_t = A_t e^{i\left(\omega t - \vec{k}_t \cdot \vec{r}\right)} = A_t e^{i\left[\omega t - k_t\left(x \sin\phi + z\cos\phi\right)\right]}$$

$$= A_t e^{i\left[\omega t - x\left(k_t \sin\phi\right) - z\left(k_t \cos\phi\right)\right]}$$

$$= A_t e^{i\left[\omega t - x\left(k_t \sin\theta / n_r\right) - z\left(\pm i\beta\right)\right]}$$

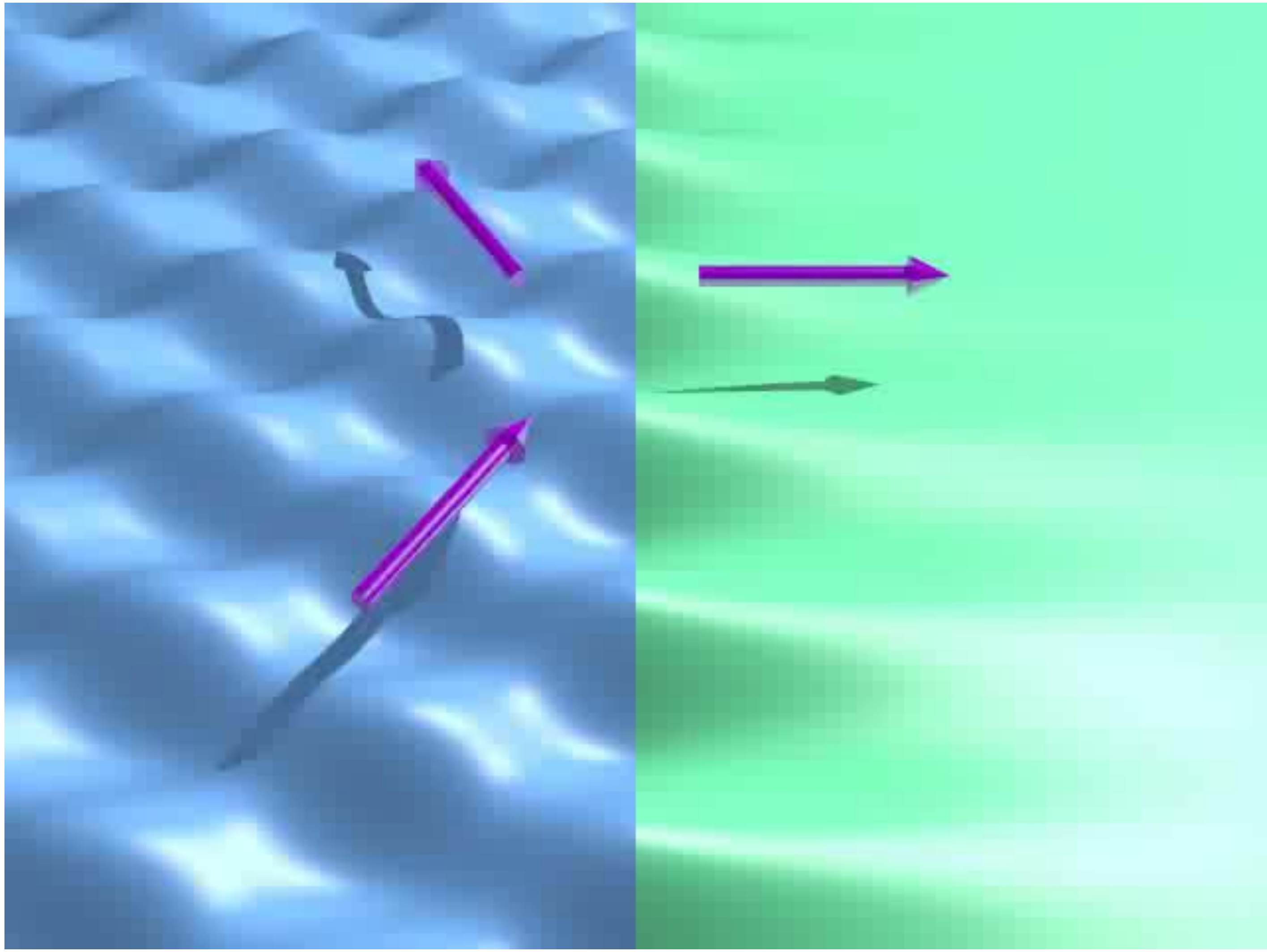
Absorption in z direction

$$E_t = A_t e^{-\beta z} e^{i\left[\omega t - x\left(k_t \sin\theta / n_r\right)\right]}$$

Propagation in x direction

Only negative sign of the amplitude exponential function has a physical meaning.

Evanescent wave



$$E_t = A_t e^{-\beta z} e^{i\left[\omega t - x\left(k_t \sin\theta / n_r\right)\right]}$$

- The disturbance travels in the x direction along the interface.
- The penetration depth depends on the refractive indices, the incident angle and the wavelength of the EM wave.

https://www.youtube.com/watch?v=KYwacKlfEpU

Frustrated total internal reflection

- If only a very thin air gap exists between two glass blocks, it is possible for energy to flow across the gap allowing the wave to propagate in the second glass block.
- The process is called **frustrated total internal reflection**.

https://www.youtube.com/watch?v=fOqF6t6A4O4

Homework # 9

Problem 9.7

An electromagnetic wave (**E**, **H**) propagates in the x-direction down a perfectly conducting hollow tube of arbitrary cross section. The tangential component of **E** at the conducting walls must be zero at all times.

Show that the solution $\mathbf{E} = E(y, z)\ \mathbf{n} \cos(\omega t - k_x x)$ substituted in the wave equation yields

$$\frac{\partial^2 E(y,z)}{\partial y^2} + \frac{\partial^2 E(y,z)}{\partial z^2} = -k^2 E(y,z),$$

where $k^2 = \omega^2/c^2 - k_x^2$ and k_x is the wave number appropriate to the x-direction, **n** is the unit vector in any direction in the (y, z) plane.

Problem 9.8

If the waveguide of Problem 9.7 is of rectangular cross-section of width a in the y-direction and height b in the z-direction, show that the boundary conditions $E_x = 0$ at $y = 0$ and a and at $z = 0$ and b in the wave equation of Problem 9.7 gives

$$E_x = A \sin\frac{m\pi y}{a} \sin\frac{n\pi z}{b} \cos(\omega t - k_x x),$$

where

$$k^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

Problem 9.9

Show, from Problems 9.7 and 9.8, that the lowest possible value of ω (the cut-off frequency) for k_x to be real is given by $m = n = 1$.